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Characterizations of Minimal $T_{3\,1/2}$ L-Topological Spaces

SHENG-GANG LI
Department of Mathematics, Shaanxi Normal University
710062, Xi'an, P.R. China
shenggangli@hotmail.com
shenggangli@21cn.com

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Abstract—For L a closed sets lattice, we prove the existence and a characterization theorem of the minimal elements of $T_{3\frac{1}{2}}LX$, where $T_{3\frac{1}{2}}LX$ is the set of all closed sets L-topologies on X ordered by inclusion.

Keywords—Fuzzy topology, Closed sets L-topology, $T_{3\frac{1}{2}}$ closed sets L-topology, Completely regular ideal, Compactness.

1. INTRODUCTION AND PRELIMINARIES

P-minimality is a meaningful topic in mathematics and its relative areas. In [1], we discussed minimal T_3 L-topological spaces and presented a method for constructing a T_3 closed set L-topology strictly weaker than a given nonminimal closed sets L-topology on X, where L is a completely distributive complete lattice. This paper is a continuation of [1], in which we will study minimal $T_{3\frac{1}{2}}$ L-topological spaces for L a more general complete lattice (called closed sets lattice). We prove the existence and a characterization theorem of minimal closed sets L-topologies on X, and we also investigate the relations of minimal $T_{3\frac{1}{2}}$ L-topological spaces and compact L-topological spaces.

Let L be a complete lattice with a smallest element 0 and a largest element 1. For every subset $K \subset L$, the infimum and supremum of K in L will be written as $\wedge K$ and $\vee K$, respectively. $e \in L - \{0\}$ is said to be a coprime element iff, for any $p, q \in L$, $e \leq p \vee q$ implies $e \leq p$ or $e \leq q$. The set of coprime elements of L will be denoted by $M(L)$. L is said to be a closed sets lattice iff, for every $a \in L$, there exists a subset $J_a \subset M(L)$ such that $a = \vee J_a$. Xu [2] proved that L is a closed sets lattice if and only if L is isomorphic to the lattice of closed sets of some topological space. Clearly, every closed sets lattice L is a coframe (i.e., L^{op} is a frame [3]), but not necessarily a completely distributive lattice (CD lattice for short).

Throughout this paper, L will be a closed sets lattice. Obviously, for every nonempty set X the set L^X of all L-subsets (i.e., the family of all mappings from X to L) is also a closed sets

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lattice under the pointwise order, and $M(L^X) = \{x_\alpha \mid x \in X,\ \alpha \in M(L)\}$, where x_α (called L point with height α) is defined by

$$x_\alpha(t) = \begin{cases} \alpha, & t = x, \\ 0, & t \neq x. \end{cases}$$

We use 0_X and 1_X to denote the smallest element and the largest element of L^X, respectively, and $\operatorname{supp} A$ the support $\{x \in X \mid A(x) > 0\}$ of an L subset $A \in L^X$. An L-topological space (L-ts for short) is a pair (X, δ), where δ, called a closed sets L-topology on X, is a subfamily of L^X closed under the operations of finite supremum and arbitrary infimum; the elements of δ are called closed elements, and for every $x_\alpha \in M(L^X)$, the set $R_\delta(x_\alpha) = \{E \in \delta \mid x_\alpha \not\leq E\}$ is called the remote neighborhoods family (cf. [4]). (X, δ) is called fully stratified iff every constant L-subset is closed.

For every ordinary mapping $f : X \longrightarrow Y$, there exists a mapping (called L-forward powerset operator, cf. [5]) $f_L^{\rightarrow} : L^X \longrightarrow L^Y$, defined by $f_L^{\rightarrow}(A)(y) = \vee\{A(x) \mid f(x) = y\}$ ($\forall A \in L^X$, $\forall y \in Y$). The right adjoint to $f_L^{\rightarrow}$ (called L-backward powerset operator, cf. [5]) is denoted by $f_L^{\leftarrow}$. It is easy to verify that $f_L^{\rightarrow}$ preserves supremum, $f_L^{\leftarrow}$ preserves supremum and infimum, and $f_L^{\leftarrow}(B) = \vee\{A \mid f_L^{\rightarrow}(A) \leq B\} = B \circ f$ ($\forall B \in L^Y$). For canonical examples of such morphisms, please see [5]. A mapping $f : (X, \delta) \longrightarrow (Y, \eta)$ (where (X, S) and (Y, η) are both L-ts) is said to be L-continuous if $f_L^{\leftarrow}(B) \in \delta$ for every $B \in \eta$. f is said to be an L-homeomorphism if it is bijective and L-continuous, and the inverse $f^{-1} : (Y, \eta) \longrightarrow (X, \delta)$ of f is L-continuous.

Let $I \subset L^X$ be an ideal (i.e., I is an up-directed lower set, cf. [1,4,6–8]) and $S = \{S(n),\ n \in D\}$ a net (i.e., a mapping $S : D \longrightarrow M(L^X)$, where D is an up-directed set) in an L-ts (X, δ), and $x_\alpha \in M(L^X)$. I is said to converge to x_α, denoted by $I \longrightarrow x_\alpha$, iff $R_\delta(x_\alpha) \subset I$; it is said to accumulate to x_α (or x_α is said to be a cluster point of I), denoted by $I \infty x_\alpha$, iff $A \vee P \neq 1_X$ for any $A \in I$ and any $P \in R_\delta(x_\alpha)$. We write $\lim I = \vee\{x_\alpha \in M(L) \mid I \longrightarrow x_\alpha\}$ and $adI = \vee\{x_\alpha \in M(L) \mid I \infty x_\alpha\}$. S is said to accumulate to x_α (or x_α is said to be a cluster point of S) iff, for any $P \in R_\delta(x_\alpha)$ and any $d \in D$, there exists an $n_d \in D$ such that $n_d \geq d$ and $S(n_d) \not\leq P$.

As in [9] (see also, [1]), two relations, $\overline{<}$ and $\overline{\leq}$, on δ may be defined as follows.

(1) $A \overline{<} B$ iff $A \wedge B^* = 0_X$, where $B^* = \wedge\{D \in \delta \mid D \vee B = 1_X\}$.

(2) $A \overline{\leq} B$ iff there exists a subset $\{C_q \mid q \in [0,1] \cap Q\} \subset \delta$ such that $A \leq C_0$, $C_1 \leq B$, and $C_p \overline{<} C_q$ for all $p, q \in [0,1] \cap Q$ satisfying $p < q$, where Q is the rational numbers set.

The separation axioms T_0, T_1, T_2, T_3, and regular of L-ts in [1] may also be extended to the case that L is a closed sets lattice.

DEFINITION 1.1. *An L-ts (X, δ) is said to be completely regular iff $A = \wedge\{B \mid B \in \delta,\ A \overline{\leq} B\}$ for every $A \in \delta$; (X, δ) is said to be $T_{3\frac{1}{2}}$ iff it is both T_0 and completely regular. An ideal $I \subset L^X$ is said to be completely regular in (X, δ) iff, for every $C \in I$, there exists a $B \in I \cap \delta$ such that $C \overline{\leq} B$.*

Obviously, every completely regular L-ts (respectively, $T_{3\frac{1}{2}}$ L-ts) is regular (respectively, T_3). These notions are also a generalization of corresponding notions in point-set topology. It is easy to verify the following three lemmas.

LEMMA 1.2. *Let (X, δ) be an L-ts and $\prec \in \{\overline{<}, \overline{\leq}\}$. Then,*

(1) $A \overline{\leq} B \Longrightarrow A \overline{<} B \Longrightarrow A \leq B$, ($\forall A, B \in \delta$).

(2) $A \leq B \prec C \leq D \Longrightarrow A \prec D$, ($\forall A, B, C, D \in \delta$).

(3) $A \prec B$ and $C \prec D \Longrightarrow A \wedge C \prec B \wedge D$ and $A \vee C \prec B \vee D$, ($\forall A, B, C, D \in \delta$).

LEMMA 1.3. *An L-ts (X, δ) is completely regular if and only if, for any $x_\alpha \in M(L^X)$ and $A \in R_\delta(x_\alpha)$, there exists a $B \in R_\delta(x_\alpha)$ such that $A \overline{\leq} B$.*

LEMMA 1.4. *Let $f : X \longrightarrow Y$ be a bijective mapping, and g the inverse mapping of f. Then, $f_L^{\rightarrow}(A) = B \Longleftrightarrow f_L^{\leftarrow}(B) = A$ ($\forall A \in L^X, \forall B \in L^Y$). Thus, $f_L^{\rightarrow} = g_L^{\leftarrow}$ and $f_L^{\leftarrow} = g_L^{\rightarrow}$.*

2. MAIN RESULTS

Analogous to [1, Theorem 2.1], we have the following theorems.

THEOREM 2.1. *For any nonempty set X, there exists a closed sets L-topology δ on X such that (X, δ) is a minimal $T_{3\frac{1}{2}}$ L-ts.*

The main result of this paper is as follows.

THEOREM 2.2. *For a $T_{3\frac{1}{2}}$ L-ts (X, δ), the following statements are equivalent.*

(1) *(X, δ) is a minimal $T_{3\frac{1}{2}}$ L-ts.*
(2) *For every $T_{3\frac{1}{2}}$ L-ts (X, σ), if the identity mapping $g : (X, \delta) \longrightarrow (X, \sigma)$ is continuous, then it is an L-homeomorphism.*
(3) *For every $T_{3\frac{1}{2}}$ L-ts (X, σ), if a mapping $f : (X, \delta) \longrightarrow (X, \sigma)$ is bijective and continuous, then it is an L-homeomorphism.*
(4) *For every $T_{3\frac{1}{2}}$ L-ts (L^Y, η), if a mapping $f : (X, \delta) \longrightarrow (Y, \eta)$ is bijective and continuous, then it is an L-homeomorphism.*
(5) *For every completely regular ideal I in (X, δ) with $|\operatorname{supp} adI| = 1$, $\lim I = adI$, where $|\operatorname{supp} adI|$ is the cardinality of the support of adI.*

PROOF. The proof of Theorem 2.2 is actually analogous to that of [1, Theorem 2.2]. For convenience of reading, we show (4) $\Longrightarrow$ (5) here.

Suppose that I is a completely regular ideal in (X, δ) with $|\operatorname{supp} adI| = 1$ and $I \infty x_\alpha \in M(L^X)$. We need to show $I \longrightarrow x_\alpha$ in (X, δ).

STEP 1. Let $\gamma = \{B \wedge C \mid B \in I \cap \delta,\ C \in R_\delta(x_\beta),\ x_\beta \in M(L^X)\}$, $\xi = (\delta - R_\delta(x_\alpha)) \cup \gamma$, and $\zeta = \{\wedge \xi_1 \mid \xi_1 \subset \xi\}$. Similar to the proof of [1, Theorem 2.2], we may verify that (X, ζ) is a T_0 L-ts.

STEP 2. (X, ζ) is $T_{3\frac{1}{2}}$. Let $y_\beta \in M(L^X)$ and $A \in R_\zeta(y_\beta)$. It suffices, by Lemma 1.3, to show that there exists a $B \in R_\zeta(y_\beta)$ such that $A \lesssim B$. Since $A \in R_\zeta(y_\beta)$, there exists a $D \in \xi$ such that $A \leq D$ and $y_\beta \not\leq D$. We consider the following two cases.

CASE 1. $D \in \delta - R_\delta(x_\alpha)$. As (X, δ) is a $T_{3\frac{1}{2}}$ L-ts, there exists a $B \in R_\delta(y_\beta)$ such that $D \lesssim B$ by Lemma 1.3. By Lemma 1.2, $A \lesssim B$. Obviously, $B \in \delta - R_\delta(x_\alpha) \subset \zeta$; i.e., $B \in R_\zeta(y_\beta)$.

CASE 2. $D \in \gamma$, i.e., $D = B_1 \wedge C_1$, where $B_1 \in I \cap \delta$, $C_1 \in R_\delta(x_t)$ for some $x_t \in M(L^X)$. As I is a completely regular ideal in (X, δ), there exists a $B_2 \in I \cap \delta$ such that $B_1 \lesssim B_2$; as (X, δ) is $T_{3\frac{1}{2}}$, there exists a $C_2 \in R_\delta(x_t)$ and $A_2 \in R_\delta(y_\beta)$ such that $C_1 \lesssim C_2$, and $A \lesssim A_2$ by Lemma 1.3. Let $B = B_2 \wedge C_2 \wedge A_2$. Then, $B \in \gamma \subset \zeta$. By Lemma 1.2, $A \lesssim B$. Apparently, $B \in R_\zeta(y_\beta)$.

STEP 3. Analogous to the proof of [1, Theorem 2.2], we may show that $\delta = \zeta$, and thus, $I \longrightarrow x_\alpha$ in (X, δ).

DEFINITION 2.3. *Let L be a CD lattice, (X, δ) an L-ts, $A \in L^X$, and $\alpha \in M(L)$. Then, there exists a subset $\beta^*(\alpha) \subset M(L)$ (cf. [4]) satisfying the following.*

(1) *$\beta^*(\alpha)$ is up-directed and $\alpha = \vee \beta^*(\alpha)$.*
(2) *For every $J \subset L$ satisfying $\alpha \leq \vee J$ and every $\lambda \in \beta^*(\alpha)$, there exists a $j_\lambda \in J$ such that $\lambda \leq j_\lambda$.*

A net $S = \{S(n), D\}$ in L^X is said to be an α-net iff, for each $\lambda \in \beta^*(\alpha)$, there exists an $n_\alpha \in D$ such that $S(n) \geq \lambda$ for all $n \in D$ satisfying $n \geq n_\alpha$. S is said to be α-net of equal height if all $S(n)$ have the same height α. Obviously, every α-net of equal height is an α-net. (X, δ) is said to be ultra-F compact [10] iff $(X, \iota_L(\delta))$ is compact, where $\iota_L(\delta)$ is a closed sets topology on X with a subbase $\{\{x \in X \mid \lambda \leq A(x)\} \mid A \in \delta,\ \lambda \in L\}$. A is said to be N-compact [8] (respectively, strong-F compact [10]) iff every α-net (respectively, every α-net of equal height) in A has a cluster point in A with height α ($\alpha \in M(L)$); it is said to be F-compact [10] iff, for every α-net S in A ($\alpha \in M(L)$) and every $\mu \in \beta^*(\alpha)$, S has a cluster point in A with height μ;

(X, δ) is said to be ultra-F compact (respectively, N-compact, strong-F compact, F-compact) iff 1_X is ultra-F compact (respectively, N-compact, strong-F compact, F-compact). Obviously, N-compact $\Longrightarrow$ strong-F compact $\Longrightarrow$ F-compact for any L-subset, and ultra-F compact $\Longrightarrow$ N-compact for any L-ts.

NOTE 2.4. It is well known that a $T_{3\frac{1}{2}}$ topological space $(X, \mathcal{J})$ is minimal if and only if it is compact (see [11,12]). However, this is not true for L-ts. Let us consider an L-ts (X, δ), where $L = \{0, a, b, 1\}$ (a and b are not comparable), $X = [0, 1]$, δ is the closed sets topology on X with a subbase $\beta = \{{}_\alpha\chi_E \mid \alpha \in L,\ \alpha \neq 0,\ X - E \in \mathcal{J}\}$, $\mathcal{J}$ is the ordinary topology on X, and ${}_\alpha\chi_E$ is defined by

$$ {}_\alpha\chi_E(x) = \begin{cases} \alpha, & x \in E, \\ 0, & x \in X - E. \end{cases} $$

We now show the following conclusions.

(1) (X, δ) is $T_{3\frac{1}{2}}$. As L is a CD lattice, for every $A \in \delta$, there exist in $(X, \mathcal{J})$ closed sets A_1, A_a, and A_b such that $A = \chi_{A_1} \vee_a \chi_{A_a} \vee_b \chi_{A_b}$ by the definition of δ. As $(X, \mathcal{J})$ is $T_{3\frac{1}{2}}$, $A_1 = \bigcap \mathcal{E}_{A_1}$, $A_a = \bigcap \mathcal{E}_{A_a}$, and $A_b = \bigcap \mathcal{E}_{A_b}$, where $\mathcal{E}_Y = \{Y' \mid X - Y' \in \mathcal{J},\ Y \precsim Y'$ in $(X, \mathcal{J})\}$ ($Y \in \{A_1, A_a, A_b\}$). Take $\mathcal{A} = \{\chi_{B_1} \vee_a \chi_{B_a} \vee_b \chi_{B_b} \mid B_1 \in \mathcal{E}_{A_1},\ B_a \in \mathcal{E}_{A_a},\ B_b \in \mathcal{E}_{A_b}\}$. Then, one may easily verify that $\mathcal{A} \subset \{B \in \delta \mid A \precsim B\}$ and $A = \wedge \mathcal{A}$, which implies that (X, δ) is completely regular. Obviously, (X, δ) is T_0.

(2) As $(X, \mathcal{J})$ is compact, (X, δ) is ultra-F compact. We will show that (X, δ) is also compact in the sense of Chang [13]. Similar to (1), for every open L-subset $A \in L^X$, there exist in $\mathcal{J}$ A_1, A_a, and A_b such that $A = \chi_{A_1} \vee_a \chi_{A_a} \vee_b \chi_{A_b}$. Suppose that β is an open cover of (X, δ). Let $\mathcal{A}_1 = \{A_1 \mid A \in \beta\}$, $\mathcal{A}_a = \{A_a \mid A \in \beta\}$, and $\mathcal{A}_b = \{A_b \mid A \in \beta\}$. Then, $\mathcal{A} = \mathcal{A}_1 \cup \{A_a \cap B_b \mid A \in \beta,\ B \in \beta\}$ is an open cover of $(X, \mathcal{J})$. Since $(X, \mathcal{J})$ is compact, there exists a finite subfamily $\beta_1 \subset \beta$ such that $\{C_1 \mid C_1 \in \beta_1\} \cup \{A_a \cap B_b \mid A \in \beta_1,\ B \in \beta_1\}$ is a cover of $(X, \mathcal{J})$. Obviously, β_1 is a cover of (X, δ); i.e., (X, δ) is compact in the sense of Chang [13].

(3) Analogous to (1), we may show that $(X, [\delta])$ is $T_{3\frac{1}{2}}$, where $[\delta] = \{A \in \delta \mid A \text{ is crisp}\}$. Therefore, (X, δ) is not minimal.

Nevertheless, the following proposition holds.

PROPOSITION 2.5. *Let L be a CD lattice, and (X, δ) a fully stratified $T_{3\frac{1}{2}}$ L-ts. If (X, δ) is F-compact, then δ is a minimal element of $\mathcal{B} = \{\xi \subset L^X \mid (X, \xi)$ is a fully stratified $T_{3\frac{1}{2}}$ L-ts$\}$.*

PROOF. Let $\xi \in \mathcal{B}$ and $\xi \subset \delta$. We need to show $\xi = \delta$. Let $f = i_X : X \longrightarrow X$ be the identity mapping of X on to itself. Then, $f : (X, \delta) \longrightarrow (Y, \xi)$ is continuous. We may verify that every closed element in an F-compact L-ts is F-compact, and that the image of an F-compact L-subset under an L-forward powerset operator induced by an L-continuous mapping is an F-compact L-subset. Thus, for each $A \in \delta$, $f_L^{\rightarrow}(A) = A$ is F-compact in (X, ξ); by [14, Lemma 2.8], $A \in \xi$. Therefore, $\xi = \delta$.

NOTE 2.6. For a fuzzy lattice (or Hutton algebra [5]) L, Hutton and Reilly [15] proposed a $T_{3\frac{1}{2}}$ separation (we call it $HRT_{3\frac{1}{2}}$) of L-ts. Analogous to [1], we may give counterexamples to show that there is no implication between $T_{3\frac{1}{2}}$ and $HRT_{3\frac{1}{2}}$.

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